Functional Analysis

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Lecture 5

Equivalence of norms.
The space of bounded operators.

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Equivalence of norms

Let us consider two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the linear space X.

Def. The norm $\|\cdot\|_1$ is **stronger** than the norm $\|\cdot\|_2$ if

$$\exists_{c>0}\ \forall_{x\in X} \qquad \|x\|_2 \leqslant c\|x\|_1 .$$

Lem. The following conditions are equivalent:

1. $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$
2. convergence in $\|\cdot\|_1$ implies convergence in $\|\cdot\|_2$
3. the topology induced by $\|\cdot\|_1$ is stronger (larger) than the topology induced by $\|\cdot\|_2$.

Proof: (1) The norm $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ $\Longleftrightarrow$ $id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is a bounded operator $\overset{\text{Thm}}{\Longleftrightarrow}$

the map $id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is continuous

(Heine's criterion) $\Updownarrow$ (2) $\qquad$ $\Updownarrow$ (topological definition) (3)

Def. The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if the norm $\|\cdot\|_1$ is both stronger and weaker than $\|\cdot\|_2$, that is, if

$$\exists_{c_1,c_2>0}\ \forall_{x\in X} \qquad \|x\|_2 \leqslant c_1\|x\|_1 \quad \text{oraz} \quad \|x\|_1 \leqslant c_2\|x\|_2.$$

Prop. The following conditions are equivalent:

1. the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent
2. convergence in $\|\cdot\|_1$ is equivalent to convergence in $\|\cdot\|_2$
3. topologies induced by $\|\cdot\|_1$ and $\|\cdot\|_2$ are equal.

Moreover, if the norms are equivalent, then,

$(X,\|\cdot\|_1)$ is a Banach space $\iff$ $(X,\|\cdot\|_2)$ is a Banach space

Proof: The equivalence of conditions (1)-(3) follows from **lem**.

If $\|\cdot\|_1$ and$\|\cdot\|_2$ are equivalent, then the sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is Cauchy in $\|\cdot\|_1$ if and only if it is Cauchy in $\|\cdot\|_2$, because

$$\|x_n - x_m\|_2 \leqslant c_1\|x_n - x_m\|_1 \quad \text{and} \quad \|x_n - x_m\|_1 \leqslant c_2\|x_n - x_m\|_2.$$

Hence equivalence of convergence yields equivalence of completeness ■

Exm. On the space $X = C[0,1]$ the following norms

$$\|x\|_\infty := \max_{t\in[0,1]} |x(t)|, \quad \|x\|_p := \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p}, \quad p \in [1,\infty)$$

are not equivalent. Indeed, take $x_n(t) = t^n$. Then $\|x_n\|_p^p = \int_0^1 t^{pn}\, dt = \frac{1}{pn+1} \to 0$, that is $x_n \xrightarrow{\|\cdot\|_p} 0$. But $\{x_n\}_{n=1}^\infty$ is not convergent in $\|\cdot\|_\infty$ (is not uniformly convergent). Hence $\|\cdot\|_p$ is not stronger, then $\|\cdot\|$. In fact,

$$\|x\|_p = \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} \leqslant \left(\int_0^1 \|x\|_\infty^p \, dt\right)^{1/p} = \|x\|_\infty$$

Hence $\|\cdot\|_\infty$ is strictly stronger than $\|\cdot\|_p$.

Uniform convergence implies convergence in L^p, but not vice versa

Ex. For $p < p'$, $\|\cdot\|_{p'}$ is strictly stronger, then $\|\cdot\|_p$

(Hint: Consider $x_n(t) = (\frac{1}{t})^{\frac{1}{p}} 1_{[\frac{1}{n},1]} + n^{\frac{1}{p}} 1_{[0,\frac{1}{n})}$ and use Hölder's ineqaulity.)

Exm. In the space $X = \mathbb{F}^n$ the norms

$$\|x\|_\infty := \max_{k=1,\dots,n} |x(k)|, \qquad \|x\|_p := \left(\sum_{k=1}^{n} |x(k)|^p\right)^{1/p}, \quad p \in [1, \infty)$$

are equivalent, because $\|x\|_\infty \leqslant \|x\|_p$ and $\|x\|_p \leqslant n\|x\|_\infty$.

Theorem.

In a finite-dimensional space all norms are equivalent.

Proof: Pick linearly independent $e_1, \dots, e_n \in X$ such that $\forall_{x \in X}\ \exists_{x(1),\dots,x(n)\in\mathbb{F}}\ x = \sum_{k=1}^{n} x(k)e_k$. Then $X \cong \mathbb{F}^n$, where $x \mapsto (x(1), \dots, x(n))$. Let us define a norm on X by the formula

$$\|x\|_\infty := \max_{k=1,\dots,n} |x(k)|, \qquad \text{where } x = \sum_{k=1}^{n} x(k)e_k.$$

It suffices to show that any norm $\|\cdot\|$ on X is equivalent to $\|\cdot\|_\infty$ (since the equivalence of norms is an equivalence relation, and in particular a transitive relation). To this end, note that

$$\|x\| = \|\sum_{k=1}^{n} x(k)e_k\| \leqslant \sum_{k=1}^{n} \|x(k)e_k\| = \sum_{k=1}^{n} |x(k)|\|e_k\| \leqslant \|x\|_\infty \sum_{k=1}^{n} \|e_k\|.$$

So $\|x\| \leqslant c\|x\|_\infty$ for $c := \sum_{k=1}^{n} \|e_k\|$. Thus $\|\cdot\|_\infty$ is stronger than $\|\cdot\|$.

Assume ad absurdum that $\|\cdot\|$ is not stronger than $\|\cdot\|_\infty$. Then for any $n \in \mathbb{N}$ there is $x_n \in X$ such that $\|x_n\|_\infty > n\|x_n\|$. Putting $y_n := \frac{x_n}{\|x_n\|_\infty}$ we get a sequence converging to 0 in $\|\cdot\|$, as

$$\|y_n\| = \frac{\|x_n\|}{\|x_n\|_\infty} < \frac{1}{n} \longrightarrow 0.$$

Hence $y_n \xrightarrow{\|\cdot\|} 0$. On the other hand, $\|y_n\|_\infty = 1$, and so $\{y_n\}_{n=1}^\infty$ is contained in the unit sphere in norm $\|\cdot\|_\infty$, which is **compact** ($\|\cdot\|_\infty$ is equivalent to the euclidean norm $\|\cdot\|_2$, see **Exm**). Hence there is a subsequence $\{y_{n_k}\}_{n=1}^\infty$ convergent in $\|\cdot\|_\infty$ to some $y \in X$ with $\|y\|_\infty = \lim_{k\to\infty} \|y_{n_k}\|_\infty = 1$. In particular $y \neq 0$. Since $\|\cdot\|_\infty$ is stronger than $\|\cdot\|$, we get $y_{n_k} \xrightarrow{\|\cdot\|} y \neq 0$  because $y_n \xrightarrow{\|\cdot\|} 0$. ∎

Cor1. Every finite-dimensional normed space is complete ($\dim(X) = n < \infty \implies X$ **is a Banach space**).

Proof: The norm on $X \cong \mathbb{F}^n$ is equivalent to the Euclidean norm. Since the Euclidean space $\mathbb{F}^n$ is complete, X is complete. ■

Cor2. A finite-dimensional subspace of normed space is closed.

Proof: Completeness implies closedness. (**Lec**.1) ■

Cor3. Any linear operator defined on a finite-dimensional space is automatically bounded (continuous).

Proof: Let $T : X \to Y$linear and $X \cong \mathbb{F}^n$. Since on X all norms are equivalent, we can assume that $\|x\| = \sum_{i=1}^n |x(k)|$, where $x = \sum_{k=1}^n x(k)e_k$. Then

$$\|Tx\| = \|\sum_{k=1}^n x(k)Te_k\| \leqslant \sum_{k=1}^n |x(k)|\|Te_k\| \leqslant C \cdot \|x\|$$

where $C := \max_{k=1,\dots,n} \|Te_k\|$. Hence T is bounded. ■

The space of bounded operators

Let X and Y be normed spaces.

The **space of bounded operators** from X to Y

$$B(X,Y) := \{T : X \to Y \text{ bounded linear}\}$$

is a normed space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$, with operations

$$(T+S)x := Tx + Sx, \qquad (\lambda T)x := \lambda Tx \quad \left(\begin{array}{c}\text{pointwise}\\ \text{operations!}\end{array}\right)$$

and the operator norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$. 

Theorem

Y is a Banach space $\iff$ $B(X,Y)$ is a Banach space

Dowód: "$\Longleftarrow$" we need the Hahn-Banach Theorem (**Lec.11**)

"$\Longrightarrow$" Let $\{T_n\}_{n=1}^{\infty} \subseteq B(X, Y)$ be Cauchy. Doe each $x \in X$

$$\|T_n x - T_m x\| \leqslant \|T_n - T_m\| \|x\| \to 0.$$

Hence the sequence $\{T_n x\}_{n=1}^{\infty} \subseteq Y$ is Cauchy in Y, hence it converges (as Y is complete). We put $Tx := \lim_{n\to\infty} T_n x$. In this way we get a linear operator $T : X \to Y$ (limes is linear!). Then

$$\begin{aligned}\|(T - T_n)x\| &= \lim_{m\to\infty} \|(T_m - T_n)x\| \\ &\leqslant \limsup_{m\to\infty} \|T_m - T_n\| \|x\| \leqslant \sup_{m\geqslant n} \|T_m - T_n\| \|x\|.\end{aligned}$$

Therefore

$$\|T - T_n\| \leqslant \sup_{m\geqslant n} \|T_m - T_n\| \longrightarrow 0, \text{ when } n \to \infty.$$

Thus $\{T_n\}_{n=1}^{\infty}$ converges in norm to T. In particular, since $T_n \in B(X, Y)$ and $T - T_n \in B(X, Y)$ (for large n), the operator $T = T - T_n + T_n \in B(X, Y)$ is bounded.

Hence $\{T_n\}_{n=1}^{\infty}$ converges in the space $B(X, Y)$. ∎

Dual space

The field of scalars $Y := \mathbb{F}$ is a Banach space.

Def.

Let X be a normed space. We call the Banach space $B(X, \mathbb{F})$ the **dual space** to X and we denote it by X^*. Elements of $X^* = B(X, \mathbb{F})$ are called bounded linear **functionals** on X.

Ex. Integration is an example of a bounded linear functional

$$L^1(\mu) \ni x \longmapsto f(x) := \int x \, d\mu \in \mathbb{F}.$$

Rem.

The main motivation for considering the dual space X^* is that it allows us to define the bilinear form (sometimes called pairing)

$$X^* \times X \ni (f, x) \longmapsto \langle f, x \rangle := f(x) \in \mathbb{F},$$

which imitates an **inner product** on X.